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# Analytic Solutions of Nonlinear Difference Equation

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## 1 Introduction

We consider the following second order nonlinear difference equation,

$$u(t+2) = f(u(t), u(t+1)), \quad (1.1)$$

where  $f$  is a holomorphic function for  $u(t), u(t+1)$ . Put  $u^*$  as a equilibrium point of (1.1). And we suppose that (1.1) has a equilibrium point  $u^* = 0$  and  $f(x, y) = -\beta x - \alpha y + g(x, y)$ , ( $\alpha, \beta$  are constants,  $\beta \neq 0$ ), where  $g$  is higher order terms for  $x, y$  such that  $g(x, y) = \sum_{i,j \geq 0, i+j \geq 2} b_{i,j} x^i y^j$ . Here we consider analytic solutions such that  $u(t) \rightarrow 0$  when  $t \rightarrow +\infty$  or  $t \rightarrow -\infty$ .

The Characteristic equation of (1.1) is

$$D(\lambda) = \lambda^2 + \alpha\lambda + \beta = 0. \quad (1.2)$$

Let  $\lambda_1, \lambda_2$  be roots of the characteristic equation and  $|\lambda_1| \leq |\lambda_2|$ . Then we consider following two case i)  $|\lambda_1| < 1$ , and ii)  $|\lambda_2| > 1$ . Of course, some characteristic equations have properties both i) and ii).

Here we consider solutions such that i)  $u(t) \rightarrow 0$ , as  $\mathbb{R}[t] \rightarrow +\infty$ , and ii)  $u(t) \rightarrow 0$ , as  $\mathbb{R}[t] \rightarrow -\infty$ .

## 2 Existence of an analytic solution

If (1.1) is a real Model, then the "t" of equation (1.1) represent "time" and  $t$  is of course a real variable. But in this section we consider  $t$  to be a complex variable, and we will prove existence of an analytic solution of (1.1) which converge to 0 with methods of complex analysis.

When we consider a real Model, after we have solutions of (1.1), we take  $t$  such as  $t \in \mathbb{R}$ . Then we can have solutions which are real values.

## 2.1 A formal solution

In case i) we put  $\lambda = \lambda_1$ , in case ii) we put  $\lambda = \lambda_2$ . Then we can define a formal solution such as

$$u(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{nt}, \quad (2.1)$$

in both cases. Where  $\alpha_1$  : arbitrary,  $\alpha_k \cdot D(\lambda^k) = C_k(\alpha_1, \dots, \alpha_{k-1})$ , ( $k = 2, \dots$ ), and  $C_k(\alpha_1, \dots, \alpha_{k-1})$  are polynomials for  $\alpha_1, \dots, \alpha_{k-1}$  with coefficients  $b_{i,j} \lambda^l$ ,  $0 \leq i \leq k$ ,  $0 \leq j \leq k$ ,  $0 \leq l \leq k$ ,  $2 \leq i+j \leq k$ . Here we suppose that  $\alpha_1 \neq 0$ .

## 2.2 Map $T$ and its Fixed Point

Here we put  $u(t) = s$ ,  $u(t+1) = w$ ,  $u(t+2) = z$ , and  $H(s, w, z) = -z + f(s, w)$ . Then the equation (1.1) can be written such as  $H(u(t), u(t+1), u(t+2)) = 0$ .

$H(s, w, z)$  is holomorphic in a neighborhood of  $(0, 0, 0)$ , and we have  $H(0, 0, 0) = 0$ , easily. Furthermore we have

$$\frac{\partial H}{\partial s}(0, 0, 0) = \frac{\partial f}{\partial s} \Big|_{s=w=0} = -\beta \neq 0.$$

So we have a holomorphic function  $\phi$  such that  $s = \phi(w, z)$  for  $|w|, |z| \leq \rho$ , (for  $\exists \rho > 0$ ). Furthermore we have a constant  $K$  such that  $|s| = |\phi(w, z)| \leq K(|w| + |z|)$  for  $|w|, |z| \leq \rho$ .

Let  $N$  be a positive integer. Put the partial sum of formal solution as  $P_N(t) = \sum_{n=1}^N \alpha_n \lambda^{nt}$ , and put  $p_N(t) = u(t) - P_N(t)$ . Here we rewrite  $p(t) = p_N(t)$ .

Moreover we define following sets,

$$S(\eta) = \{t \in \mathbb{C} : |\lambda^t| \leq \eta\}$$

$$J(A, \eta) = \{p : p(t) \text{ is holomorphic and } |p(t)| \leq A|\lambda^t|^{N+1} \text{ for } t \in S(\eta)\}.$$

in which  $A > 0$  and  $\eta$ ,  $0 < \eta < 1$ , are constants to be determined later.

### 2.2.1 The case i) $|\lambda| < 1$

In this case, our aim is to prove the existence of  $u(t)$  when  $\mathbb{R}[t] \rightarrow \infty$ , such that

$$u(t) = \phi(u(t+1), u(t+2)).$$

If we have the analytic solution  $u(t)$ , then it is the solution of (1.1), and have a solution  $p$  of following equation,

$$p(t) = \phi(p(t+1) + P_N(t+1), p(t+2) + P_N(t+2)) - P_N(t).$$

Conversely if  $p(t)$  which satisfies above equation would exist, then we have a solution  $u(t)$  of (1.1) which has the expansion (2.1) by  $u(t) = p(t) + P_N(t)$ .

For  $p(t) \in J(A, \eta)$ , put

$$T_1[p](t) = \phi(p(t+1) + P_N(t+1), p(t+2) + P_N(t+2)) - P_N(t).$$

**Lemma 1.** *We have a fixed point  $p(t) = p_N(t) \in J(A, \eta)$  of  $T_1$ , which depends on  $N$ .*

**Proof.** Since  $\phi$  is holomorphic on  $|w| \leq \rho$ ,  $|z| \leq \rho$  we have

$$\left| \frac{\partial \phi}{\partial w} \right|, \left| \frac{\partial \phi}{\partial z} \right| \leq \frac{8K}{\rho} \quad \text{for } |w|, |z| \leq \frac{\rho}{2}.$$

Next we take  $A$ , and take  $\eta$  sufficiently small such that  $A\eta^{N+1} < \frac{\rho}{4}$ . Then for sufficiently large  $t$ , we have  $|p(t)| \leq A|\lambda^t|^{N+1} < \frac{\rho}{4}$ ,  $|p(t+1)| \leq A|\lambda|^{N+1}|\lambda^t|^{N+1} < \frac{\rho}{4}$ ,  $|p(t+2)| \leq A|\lambda|^{2(N+1)}|\lambda^t|^{N+1} < \frac{\rho}{4}$ . Furthermore we can obtain  $|w|, |z| \leq \frac{\rho}{2}$ . So we have

$$|T_1[p](t)| \leq \left( \frac{16K}{\rho} A|\lambda|^{N+1} + K_2 \right) |\lambda^t|^{N+1}. \quad (2.2)$$

where  $K_2$  is constant, depends on  $N$ . Hence we have If we suppose  $N$  is so large that  $\frac{16K}{\rho}|\lambda|^{N+1} < \frac{1}{4}$ , furthermore we take  $A$  so large that  $A > \frac{4}{3}K_2$ , then

$$|T_1[p](t)| < A|\lambda^t|^{N+1}.$$

So we obtain that  $T_1$  maps  $J(A, \eta)$  into itself, The map  $T_1$  is continuous if  $J(A, \eta)$  is endowed with topology of uniform convergence on compact set in  $S(\eta)$ , and  $J(A, \eta)$  is convex, and is relatively compact set.

Thus by Schauder's fixed point theorem in [2], we obtain the existence of a fixed point  $p(t) = p_N(t) \in J(A, \eta)$  of  $T_1$ .  $\square$

### 2.2.2 The case ii) $|\lambda| > 1$

In this case, our aim is to prove the existence of  $u(t)$  when  $\mathbb{R}[t] \rightarrow \infty$ , such that  $u(t) = f(u(t-2), u(t-1))$ .

If we have an the analytic solution  $u(t)$ , then it is the solution of (1.1). And we have a solution  $p$  of following equation,

$$p(t) = f(p(t-2) + P_N(t-2), p(t-1) + P_N(t-1)) - P_N(t).$$

Conversely if  $p(t)$  which satisfies above equation would exist, then we have a solution  $u(t)$  of (1.1) which has the expansion (2.1) by  $u(t) = p(t) + P_N(t)$ .

For  $p(t) \in J(A, \eta)$ , put

$$T_2[p](t) = f(p(t-2) + P_N(t-2), p(t-1) + P_N(t-1)) - P_N(t).$$

**Lemma 2.** We have a fixed point  $p(t) = p_N(t) \in J(A, \eta)$  of  $T_2$ , which depends on  $N$ .

**Proof.** Here we put  $s = u(t-2)$ ,  $w = u(t-1)$ ,  $z = u(t)$ . Since  $f$  is holomorphic on  $|s| \leq \rho$ ,  $|w| \leq \rho$  we have Hence we have

$$\left| \frac{\partial f}{\partial s} \right|, \left| \frac{\partial f}{\partial w} \right| \leq \frac{8K_1}{\rho} \quad \text{for } |s|, |w| \leq \frac{\rho}{2},$$

where  $K_1$  is a constant. Next we take  $A$ , and take  $\eta$  sufficiently small such that  $A\eta^{N+1} < \frac{\rho}{4}$ .

Then for sufficiently large  $-t$ , we have

$$|T_2[p](t)| \leq \left( \frac{16K_1}{\rho} A |\lambda|^{-(N+1)} + K_3 \right) |\lambda^t|^{N+1}.$$

with a constant  $K_3$  which depends on  $N$ .

If we suppose  $N$  is so large that  $\frac{16K_1}{\rho} |\lambda|^{N+1} < \frac{1}{4}$ , and we take  $A$  so large that  $A > \frac{4}{3} K_3$ , then

$$|T_2[p](t)| < A |\lambda^t|^{N+1}.$$

So we obtain that  $T_2$  maps  $J(A, \eta)$  into itself,  $T_2$  maps  $J(A, \eta)$  into itself, The map  $T_2$  is continuous if  $J(A, \eta)$  is endowed with topology of uniform convergence on compact set in  $S(\eta)$ , and  $J(A, \eta)$  is convex, and is relatively compact set.

Thus by Schauder's fixed point theorem in [3], we We obtain the existence of a fixed point  $p(t) = p_N(t) \in J(A, \eta)$  of  $T_2$ .  $\square$

## 2.3 Uniqueness of the Fixed Point

We can have following two lemmas.

**Lemma 3.** The fixed point  $p_N(t) \in J(A, \eta)$  of  $T_1$  is unique for each  $N$ .

**Lemma 4.** The fixed point  $p_N(t) \in J(A, \eta)$  of  $T_2$  is unique for each  $N$ .

## 2.4 Proof that the solution $u(t) = p_N(t) + P_N(t)$ is independent of $N$

Finally we will show that the solution  $u(t)$ , given by  $u(t) = p_N(t) + P_N(t)$  does not depend on  $N$ . Then we obtain that (2.1) gives an exact solution of (1.1).

**Lemma 5.** The solution  $u_N(t) = p_N(t) + P_N(t)$  of (1.1) is independent of  $N$ .

## 2.5 the analytic solution $u(t)$ of (1.1)

From lemma 1-lemma 5, we have proved that a solution  $u(t)$  is defined and holomorphic in  $S(\eta)$  for a  $\eta > 0$ , which has the expansion  $u(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{nt}$ . Hence we have the following Theorem 6.

**Theorem 6.** Let  $\lambda_1, \lambda_2$  be roots of the characteristic equation of (1.1) and  $|\lambda_1| \leq |\lambda_2|$ . If  $|\lambda_1| < 1$  or  $|\lambda_2| > 1$ , then we have the holomorphic solution  $u(t)$  of (1.1) in  $S(\eta)$  for a  $\eta(> 0)$ , which has the expansion  $u(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{nt}$ .

However, we cannot assume the condition  $\frac{\partial H}{\partial s}(s, w, z) \neq 0$ , for all  $s$  in case i), if  $\frac{\partial H}{\partial s}(s, w, z) = 0$ , for some,  $w, z$ , then the  $(w, z)$  are branch points. The solution  $u(t)$  can be continued analytically by making use of the relation

$$u(t-2) = \phi(u(t-1), u(t)),$$

keeping out of branch points, up to  $\mathbb{R}[t] \geq 0$ . The solution obtained may be multivalued.

### 3 Analytic General Solutions

**Theorem 7.** Suppose that  $u(\tau)$  is the solution of (1.1) which we have in Theorem 6, and has the expansion  $u(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{nt}$ . Further suppose that  $\chi(t)$  is an analytic solution of (1.1) such that  $\chi(t+n) \rightarrow 0$  as when  $\lambda < 1$ ,  $n \rightarrow +\infty$ , and as when  $\lambda > 1$ ,  $n \rightarrow -\infty$  uniformly on any compact set.

Then there is a periodic entire function  $\pi(t)$ , ( $\pi(t+1) = \pi(t)$ ), such that

$$\chi(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{n(\frac{\log \pi(t)}{\log \lambda} + t)} = \sum_{n=1}^{\infty} \alpha_n \pi(t)^n \lambda^{nt},$$

where  $\pi(t)$  is an arbitrarily periodic function whose period is one.

Conversely, if we put

$$\chi(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{n(\frac{\log \pi(t)}{\log \lambda} + t)} = \sum_{n=1}^{\infty} \alpha_n \pi(t)^n \lambda^{nt},$$

where  $\pi$  is a periodic function whose period is one, then  $\chi(t)$  is a solution of (1.1).

**Proof.** Here we prove in the case  $\lambda < 1$ .

Let  $u(\tau)$  be the solution of (1.1) in above argument. And suppose  $\chi(t)$  be a solution of (1.1) such that  $\chi(t+n) \rightarrow 0$  as  $n \rightarrow +\infty$  uniformly on any compact set.

We put

$$u(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{nt} = U(\lambda^t), \quad \alpha_1 \neq 0,$$

then  $U, \chi$  are open maps, and  $U(0) = 0$ . So we have  $\chi(t) = U(\tau) = U(\lambda^\sigma)$  (for  $\exists \tau = \lambda^\sigma$ ). Since  $\alpha_1 \neq 0$ , we have  $\sigma = \log_\lambda U^{-1}(\chi(t)) := l(t)$ .

Here according to [3], ([5]), we can prove existence of  $\Psi$  such that

$$\Psi(F(\chi, \Psi(\chi))) = G(\chi, \Psi(\chi)),$$

where  $F(s, w) = w$ ,  $G(s, w) = f(s, w)$ . Then we obtain the following first order difference equation from (1.1)

$$\chi(t+1) = \Psi(\chi(t)).$$

And we obtain

$$l(t) = t + \pi(t) \quad (\pi : \text{arbitrarily period one}).$$

Now we put  $\lambda^{\pi(t)}$  into  $\pi(t)$ . Then  $\chi(t)$  can be written as

$$\chi(t) = \sum_{n=1}^{\infty} \alpha_n \lambda^{n(\frac{\log \pi(t)}{\log \lambda} + t)} = \sum_{n=1}^{\infty} \alpha_n \pi(t)^n \lambda^{nt},$$

where  $\pi$  is an arbitrarily periodic function whose period is one.  $\square$

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